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# Eulerian dynamics with a commutator forcing

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We study a general class of Euler equations driven by a forcing with a *commutator structure* of the form $[\mathcal{L}, \mathbf{u}](\rho) = \mathcal{L}(\rho\mathbf{u}) - \mathcal{L}(\rho)\mathbf{u}$, where $\mathbf{u}$ is the velocity field and $\mathcal{L}$ is the 'action' which belongs to a rather general class of translation invariant operators. Such systems arise, for example, as the hydrodynamic description of velocity alignment, where action involves convolutions with bounded, positive influence kernels, $\mathcal{L}_\phi(f) = \phi * f$. Our interest lies with a much larger class of $\mathcal{L}$'s which are neither bounded nor positive.

In this article, we develop a global regularity theory in the one-dimensional setting, considering three prototypical subclasses of actions. We prove global regularity for *bounded* $\phi$'s which otherwise are allowed to change sign. Here we derive sharp critical thresholds such that sub-critical initial data $(\rho_0, u_0)$ give rise to global smooth solutions. Next, we study *singular* actions associated with $\mathcal{L} = -(-\partial_{xx})^{\alpha/2}$, which embed the fractional Burgers' equation of order $\alpha$. We prove global regularity for $\alpha \in [1, 2)$. Interestingly, the singularity of the fractional kernel $|x|^{-(1+\alpha)}$, avoids an initial threshold restriction. Global regularity of the critical endpoint $\alpha = 1$ follows with double-exponential $W^{1,\infty}$-bounds. Finally, for the other endpoint $\alpha = 2$, we prove the global regularity of the Navier–Stokes equations with density-dependent viscosity associated with the *local* $\mathcal{L} = \partial_{xx}$.

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## 1. Fundamentals Euler equations with a commutator structure

We are concerned with a new class of Eulerian dynamics where a velocity field, $\mathbf{u} : \Omega \times \mathbb{R}_+ \mapsto \mathbb{R}^n$, is driven by the system

$$\begin{cases} \rho_t + \nabla \cdot (\rho\mathbf{u}) = 0, \\ \mathbf{u}_t + \mathbf{u} \cdot \nabla\mathbf{u} = \mathcal{T}(\rho, \mathbf{u}), \end{cases} \qquad (\mathbf{x}, t) \in \Omega \times \mathbb{R}_+. \tag{1.1}$$

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The main feature here is the commutator structure of the forcing

$$\mathcal{T}(\rho,\mathbf{u}) = [\mathcal{L},\mathbf{u}](\rho) := \mathcal{L}(\rho\mathbf{u}) - \mathcal{L}(\rho)\mathbf{u} \tag{1.2}$$

expressed in terms of a self-adjoint operator $\mathcal{L} : \mathbb{R} \mapsto \mathbb{R}$ (the action on $\rho\mathbf{u}$ is interpreted component wise). We focus on the Cauchy problem over the whole space $\Omega = \mathbb{R}^n$ or over the torus $\Omega = \mathbb{T}^n$.

A typical example is provided by radial mollifiers, $\mathcal{L}(f) = \phi * f$, associated with integrable $\phi \in L^1$, which yields the commutator forcing

$$\mathcal{T}(\rho,\mathbf{u})(\mathbf{x}) = \phi * (\rho\mathbf{u}) - (\phi * \rho)\mathbf{u} = \int_{\mathbb{R}^n} \phi(|\mathbf{x}-\mathbf{y}|)(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x}))\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}. \tag{1.3}$$

The corresponding system (1.1) and (1.3) arises as macroscopic realization of the Cucker–Smale agent-based dynamics (Cucker & Smale, 2007a,b), which describes the collective motion of $N$ agents, each of which adjusts its velocity to a weighted average of velocities of its neighbors dictated by an *influence function* $\phi$,

$$\begin{cases} \dot{\mathbf{x}}_i &= \mathbf{v}_i, \\ \dot{\mathbf{v}}_i &= \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \phi(|\mathbf{x}_i - \mathbf{x}_j|)(\mathbf{v}_j - \mathbf{v}_i), \end{cases} \qquad (\mathbf{x}_i,\mathbf{v}_i) \in \mathbb{R}^n \times \mathbb{R}^n.$$

For large crowds, $N \gg 1$, one is led to the pressureless hydrodynamic description (1.1) and (1.3), under a mono-kinetic ansatz (Ha & Tadmor, 2008; Carrillo *et al.*, 2017). For recent results which justify the passage to Cucker–Smale kinetic and hydrodynamic descriptions with *weakly singular* kernels $\phi$ (of order $< \frac{1}{2}$) we refer to Peszek (2015) and Poyato & Soler (2016). The global regularity of such one- and two-dimensional systems (1.1) and (1.3) was studied in Tadmor & Tan (2014), Carrillo *et al.* (2016) and He & Tadmor (2016). For *bounded, positive* mollifiers it was shown that there exist certain *critical thresholds* in the phase space of initial configurations, $(\rho_0 > 0, \mathbf{u}_0)$, such that sub-critical initial data propagate the initial smoothness of $(\rho(\cdot,0),\mathbf{u}(\cdot,0)) = (\rho_0,\mathbf{u}_0)$ globally in time.

Our interest lies in the global regularity of (1.1) and (1.2) for a much larger class of $\mathcal{L}$'s which are neither positive nor bounded. We have three typical examples in mind.

### 1.1. *Examples*

Consider $\mathcal{L} = \mathcal{L}_\phi$ of the form

$$\mathcal{L}_\phi(f)(\mathbf{x}) := \int_{\mathbb{R}^n} \phi(|\mathbf{x}-\mathbf{y}|)\big(f(\mathbf{y}) - f(\mathbf{x})\big)\,\mathrm{d}\mathbf{y}. \tag{1.4}$$

Our first example involves *bounded* kernels with a finite positive mass, denoted $\phi \in L^\infty_\# := \{\phi \in L^\infty \,|\, 0 < \int \phi(r)\,\mathrm{d}r < \infty\}$, but otherwise are allowed to *change sign*. The resulting commutator $\mathcal{T}_\phi = [\mathcal{L}_\phi,\mathbf{u}](\rho)$ coincides with the usual convolution action in (1.3),

$$\mathcal{T}(\rho,\mathbf{u})(\mathbf{x}) = [\mathcal{L}_\phi,\mathbf{u}](\rho)(\mathbf{x}) = \int_{\mathbb{R}^n} \phi(|\mathbf{x}-\mathbf{y}|)(\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x}))\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}, \qquad \phi \in L^\infty_\#. \tag{1.5}$$

The action $\mathcal{L}_\phi$ in (1.4) and its commutator forcing (1.5) are well defined for *nonintegrable* $\phi$'s as well. As a second example we consider, $\phi_\alpha(\mathbf{x}) := |\mathbf{x}|^{-(n+\alpha)}$, associated with the action of the fractional Laplacian[†] $\mathcal{L}_\alpha(f) = -\Lambda_\alpha(f)$, $\alpha < 2$,

$$\Lambda_\alpha(f)(\mathbf{x}) = p.v. \int_{\mathbb{R}^n} \frac{f(\mathbf{x}) - f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{n+\alpha}} \, \mathrm{d}\mathbf{y}, \qquad \Lambda_\alpha = (-\Delta)^{\alpha/2}, \quad 0 < \alpha < 2.$$

The corresponding forcing is then given by the singular integral

$$\mathcal{T}(\rho, \mathbf{u})(x) = -\Lambda_\alpha(\rho\mathbf{u}) + \Lambda_\alpha(\rho)\mathbf{u} = p.v. \int_{\mathbb{R}^n} \frac{\mathbf{u}(\mathbf{y}) - \mathbf{u}(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^{n+\alpha}} \rho(\mathbf{y}) \, \mathrm{d}\mathbf{y}. \tag{1.6}$$

The operator $\mathcal{T}$ in (1.6) is well defined as a distribution over the whole space $\Omega = \mathbb{R}^n$. When dealing with the torus $\Omega = \mathbb{T}^n$, the forcing $\mathcal{T}$ can be expressed in terms of the periodized kernel $\phi_\alpha(\mathbf{z}) = \sum_{\mathbf{k}\in\mathbb{Z}^n} \frac{1}{|\mathbf{z}+2\pi\mathbf{k}|^{n+\alpha}}$.

Finally, as a third example we consider the full Laplacian $\mathcal{L} = \Delta$ corresponding to the limiting case $\alpha = 2$ with forcing $\mathcal{T}(\rho, \mathbf{u}) = \rho\Delta\mathbf{u} + 2(\nabla\rho\cdot\nabla)\mathbf{u}$. This leads to the density-dependent system of pressureless Navier–Stokes equations

$$\begin{cases} \rho_t + \nabla\cdot(\rho\mathbf{u}) = 0, \\ (\rho\mathbf{u})_t + \nabla(\rho\mathbf{u}\otimes\mathbf{u}) = \nabla(\rho^2 D\mathbf{u}), \qquad D\mathbf{u} = \{\partial_i u_j\}. \end{cases} \tag{1.7}$$

We close by noting that these equations are typically come 'equipped' with certain standard global bounds. Thus, in addition to the obvious conservation of mass,

$$M_0 := \int \rho_0(\mathbf{x}) d\mathbf{x} \equiv \int \rho(\mathbf{x}, t) \, \mathrm{d}\mathbf{x},$$

we have, since $\mathcal{L}$ is assumed self-adjoint, $\int \big(\rho\mathcal{L}(\rho\mathbf{u}) - \mathcal{L}(\rho)\rho\mathbf{u}\big) \, \mathrm{d}\mathbf{x} = 0$, conservation of momentum, $\int_{\mathbb{R}^n} \rho\mathbf{u}(\cdot, t) \, \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n} \rho_0\mathbf{u}_0 \, \mathrm{d}\mathbf{x}$. Also for $\mathcal{L}_\phi$ we have the $\rho$-weighted energy-enstrophy bound

$$\int_{\mathbb{R}^n\times\{T\}} \rho|\mathbf{u}|^2 \, \mathrm{d}\mathbf{x} + \int_0^T \int_{\mathbb{R}^n\times\mathbb{R}^n} \rho(\mathbf{x})\rho(\mathbf{y})\phi(|\mathbf{x}-\mathbf{y}|)|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})|^2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{y} \, \mathrm{d}t = \int_{\mathbb{R}^n} \rho_0|\mathbf{u}_0|^2 \, \mathrm{d}\mathbf{x}. \tag{1.8}$$

### 1.2. *The one-dimensional case. Statement of main results*

The main focus of this article is one-dimensional case where (1.1) and (1.2) read,

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = \rho\mathcal{L}(\rho u) - \rho\mathcal{L}(\rho)u, \end{cases} \qquad (x, t) \in \Omega \times \mathbb{R}_+. \tag{1.9}$$

We shall make a detailed study on the propagation of regularity of (1.9) for sub-critical initial data, dictated by the properties of $\mathcal{L}$.

[†] We shall abuse notations by abbreviating $\mathcal{L}_{\phi_\alpha} := \mathcal{L}_\alpha$ since the distinction is clear from the context of the sub-index involved.

We begin by recalling that (1.9) with $\mathcal{L} = \mathcal{L}_\phi$ amounts to the one-dimensional Cucker–Smale 'flocking hydrodynamics' (Cucker & Smale, 2007a,b; Motsch & Tadmor, 2014; Carrillo *et al.*, 2017)

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = \displaystyle\int_{\mathbb{R}} \phi(|x-y|)\big(u(y) - u(x)\big)\rho(x)\rho(y)\,\mathrm{d}y, \end{cases} \qquad (x,t) \in \Omega \times \mathbb{R}_+. \tag{1.10}$$

Global regularity for bounded *positive* $\phi$'s persists if and only if the initial data are *sub-critical* in the sense that (Carrillo *et al.*, 2016)

$$u_0'(x) + \phi * \rho_0(x) \geqslant 0 \quad \text{for all } \ x \in \mathbb{R}. \tag{1.11}$$

In Section 3.1, we extend this regularity result for general bounded $\phi$'s whether positive or not. The results below are stated over the torus, $\Omega = \mathbb{T}^1$, for the purely technical reason of securing a *uniform* lower bound of the density away from vacuum, which in turn provides uniform parabolicity of the $u$-equation. However, the local well-posedness follows from our analysis over $\Omega = \mathbb{R}$ line as well.

THEOREM 1.1 Consider the hydrodynamics flocking model (1.10) with a bounded mollifier, $\phi \in L^\infty_\#$ having a positive total mass $I(\phi) = \int \phi(r)dr > 0$, and subject to sub-critical initial data $(\rho_0, u_0) \in (L^1_+(\mathbb{T}^1), W^{1,\infty}(\mathbb{T}^1))$, such that

$$u_0'(x) + \phi * \rho_0(x) > 0, \qquad x \in \mathbb{T}^1.$$

Then (1.10) admits global smooth solution.

Next, we extend this result to the case of the positive *singular* mollifiers $\phi_\alpha(r) = |r|^{-(1+\alpha)}$

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = p.v. \displaystyle\int_{\Omega} \frac{u(y) - u(x)}{|x-y|^{1+\alpha}}\rho(x)\rho(y)\,\mathrm{d}y, \quad \alpha < 2, \end{cases} \qquad (x,t) \in \Omega \times \mathbb{R}_+. \tag{1.12}$$

Here we follow a general iteration scheme for proving (higher) regularity outlined in Section 2, in which one seeks bounds on the density, $\rho$, and then bounds the 'action' $\mathcal{L}(\rho)$. The uniform bounds on the density for all three cases are worked out in Section 3. We then turn to secure bounds on the action or—what amounts to the same thing, uniform bound on $u_x$, which in turn yields global well-posedness. In Section 4, we discuss the global regularity for bounded mollifiers $\mathcal{L} = \mathcal{L}_\phi$ in (1.10), and in Section 5 for the Navier–Stokes equations, $\mathcal{L} = \Delta$,

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = (\rho^2 u_x)_x, \end{cases} \qquad (x,t) \in \Omega \times \mathbb{R}_+. \tag{1.13}$$

This is the one-dimensional special case of the general class of Navier–Stokes equations studied in Bresch *et al.* (2007).

THEOREM 1.2 Consider the Navier–Stokes equations (1.13) subject to initial data $(u_0, \rho_0) \in H^2(\mathbb{T}^1) \times H^3(\mathbb{T}^1)$. Then (1.13) admits a global solution in the same class.

Finally, in Section 6 we prove the global smooth solutions for the commutator forcing associated with the *singular* action $\mathcal{L}_\alpha(\rho) = -\Lambda_\alpha(\rho)$ corresponding to singular kernel $\phi_\alpha(r) = |r|^{-(1+\alpha)}$, $1 \leqslant \alpha < 2$.

THEOREM 1.3 Consider the system of equations (1.12) with $1 \leqslant \alpha < 2$ subject to initial data $(u_0, \rho_0) \in H^3(\mathbb{T}^1) \times H^{2+\alpha}(\mathbb{T}^1)$. Then (1.12) admits a global solution in the same class.

It is remarkable that the singularity of $\phi_\alpha = |x|^{-(1+\alpha)}$ removes the requirement for a finite critical threshold which is otherwise called for integrable $\phi \in L^\infty_\#$. Specifically, in Section 3.2 we prove that for *any* singular kernel such that $\lim_{r\downarrow 0[\mathrm{mod}\, 2\pi]} r \cdot \min_{|z|\leqslant r} \phi(|z|) \uparrow \infty$, the density of the corresponding system (1.10) remains *uniformly* bounded which in turn drives the global regularity. The analysis of equation with the singular action $\mathcal{L}_\alpha$ becomes critical when $\alpha$ value reaches 1. The necessary $W^{1,\infty}$-bounds on the solution pair $(u, \rho)$ in this case admit double-exponential growth in time, consult (6.22).

REMARK 1.4 (On the singular case of fractional order $\alpha \in (0, 1)$). Recently, shortly after the release of our results at arXiv:1612.04297, we learned of another approach to the regularity of (1.12) with singular kernels $\phi_\alpha$ of order $\alpha \in (0, 1)$ that appeared in the work of Do *et al.* (2017). In their alternative approach, based on a conservation law for a first-order quantity as in Section 2.2 below and propagating a modulus of continuity adapted to the problem at hand, $(1.12)_\alpha$ is treated as critical system for the *full* range $\alpha \in (0, 1)$. However, the case $\alpha = 1$ appears to be a 'critical barrier' for passing to the range $\alpha \in [1, 2)$ treated in this article.

## 2. Propagation of global regularity A general iteration scheme

### 2.1. $L^\infty$-*bound of the velocity*

We assume that $\mathcal{L}$ satisfies the following *monotonicity condition*. Let $x_+ = \arg\max_x g(x)$ and $x_- = \arg\min_x g(x)$. Then for $f \geqslant 0$

$$\begin{cases} \mathcal{L}(fg)(x_+) \leqslant \mathcal{L}(f)(x_+)g(x_+), & g(x_+) = \max_x g(x) \\ \mathcal{L}(fg)(x_-) \geqslant \mathcal{L}(f)(x_-)g(x_-), & g(x_-) = \min_x g(x) \end{cases} \tag{2.1}$$

which holds for $\mathcal{L} = \mathcal{L}_\phi$ with positive $\phi$'s. Application of (2.1) with $(f, g) = (\rho, u)$ implies that

$$\mathcal{T}(\rho, u)(x_+) \leqslant 0 \leqslant \mathcal{T}(\rho, u)(x_-), \qquad x_\pm = \begin{cases} \arg\max\ u(\cdot, t) \\ \arg\min\ u(\cdot, t) \end{cases}$$

and yields that $u$ in (1.9) (and likewise—the velocity components $u_i$ in (1.1)) satisfy maximum/minimum principle

$$\min_{\mathbf{x}} u_i(\mathbf{x}, 0) \leqslant u_i(\mathbf{x}, t) \leqslant \max_{\mathbf{x}} u_i(\mathbf{x}, 0). \tag{2.2}$$

Likewise, $\|\mathbf{u}(\cdot, t)\|_{L^\infty}$ remains finite for $\phi \in L^\infty_\#$ with arbitrary sign.

## 2.2. *Critical threshold and a first order conservation law*

We outline our general strategy for tracing the global regularity of (1.9). The key observation is that the commutator form of (1.9) entails the transport of $u_x + \mathcal{L}(\rho)$ away from vacuum. To this end, differentiate (1.1) to find that $u' := u_x$ satisfies

$$u'_t + uu'_x + (u')^2 = \mathcal{L}(\rho u)_x - u\mathcal{L}(\rho)_x - u'\mathcal{L}(\rho). \tag{2.3}$$

For the latter we use the density equation, $\mathcal{L}(\rho u)_x = \mathcal{L}\big((\rho u)_x\big) = -\mathcal{L}(\rho)_t$ to conclude

$$(u' + \mathcal{L}(\rho))_t + u(u' + \mathcal{L}(\rho))_x + u'(u' + \mathcal{L}(\rho)) = 0.$$

This calls for introduction of the new variable,[2] $e := u' + \mathcal{L}(\rho)$, which is found to satisfy

$$e_t + (ue)_x = 0, \qquad e = u' + \mathcal{L}(\rho). \tag{2.4}$$

Together with the density equation, this yields that $e/\rho$ is governed by the transport equation

$$\left(\frac{e}{\rho}\right)_t + u\left(\frac{e}{\rho}\right)_x = 0. \tag{2.5}$$

Hence $e/\rho$ remains constant along the characteristics $\dot{x}(t) = u(x(t), t)$,

$$\frac{e(x(t), t)}{\rho(x(t), t)} = \frac{e_0(x)}{\rho_0(x)}. \tag{2.6}$$

It follows that if $e_0/\rho_0$ is allowed to have singularities, then these initial singularities will propagate along characteristics and a solution of (1.9) will consist of strips of regularity trapped between the curves carrying these singularities. To avoid this scenario, calls for the following bound to hold.

Assumption 2.1 [Critical threshold] There exist finite constants $\eta_- \leqslant 0 < \eta_+$ such that

$$\eta_- \leqslant \frac{e_0(x)}{\rho_0(x)} \leqslant \eta_+ \ \text{ for all } \ x \in \Omega. \tag{2.7}$$

Remark 2.2 We note in passing that integration of (2.7) yields $\eta_- M_0 \leqslant \int \big(u'_0 + \mathcal{L}(\rho_0)\big)\,\mathrm{d}x$. Hence, since $\mathcal{L}_\phi(\rho_0)$ has zero mean and $u(\cdot, t)$ is either periodic or assumed to have vanishing far-field boundary values, it follows that (2.7) requires $\eta_- \leqslant 0$.

We will investigate the propagation of regularity of solutions subject to sub-critical initial data (2.7).

[†] The distinction between the variable '$e$' and the usual number '$e$' will be clear from the text.

### 2.3. *The iteration scheme—a priori control estimates via e*

The study of global well-posedness for all three cases of commutator forcing we have in mind—bounded, singular and local (NS) mollifiers, share a common scheme of establishing control over the key quantities, even though the handling of the three cases is quite different when it comes to analytic details. In this section, we highlight those main common features in three steps.

• **Step** #1 (*Pointwise bounds on the density*). Our aim is to show that for a certain range of threshold bounds $\eta_- \leqslant 0 < \eta_+$, the density remains bounded from above and away from the vacuum

$$0 < \rho_- \leqslant \rho(\cdot, t) \leqslant \rho_+ < \infty. \tag{2.8}$$

In view of transportation of the ratio $e/\rho$, (2.6), we also have

$$\eta_- \leqslant \frac{e(\cdot, t)}{\rho(\cdot, t)} \leqslant \eta_+, \qquad \eta_- \leqslant 0. \tag{2.9}$$

We conclude that the quantity of interest, $e = u_x + \mathcal{L}(\rho)$, will remain uniformly bounded,

$$e_- := \eta_-\rho_+ \leqslant e(\cdot, t) \leqslant e_+ := \eta_+\rho_+.$$

• **Step** #2 (*Pointwise bound on the action* $\mathcal{L}(\rho)$ *and slope* $u_x$). Equipped with the uniform bound on $e$ we turn to establish a bound on the action $\mathcal{L}(\rho)$, which is equivalent to controlling the slope $u_x$. In the case of bounded mollifiers, we seek a point-wise bound on the action $\mathcal{L}(\rho)$

$$\mathcal{L}_- \leqslant \mathcal{L}(\rho) \leqslant \mathcal{L}_+, \qquad \rho \in L^1_+ \cap L^\infty. \tag{2.10}$$

This will imply the desired $C^1$-bound of the velocity

$$\eta_-\rho_+ - \mathcal{L}_+ \leqslant u_x(\cdot, t) \leqslant \eta_+\rho_+ + \mathcal{L}_-.$$

For singular fractional mollifiers $\mathcal{L}_\alpha$, we focus on the critical case $\alpha = 1$, where we use a nonlocal maximum principle to establish control over $\rho'$ which in turn enables us to control $u_x$ indirectly, thus avoiding an additional obstacle coming from the Hilbert transform. For the NS case, we first control the slope $u_x$ via energy bounds, then conclude with control of $\mathcal{L}(\rho) = \rho_{xx}$.

It is clear from the fact that the higher-order quantity $e$ satisfies lower-order estimates that a proper statement of well-posendess result for singular mollifiers requires $\rho$ to be in a regularity class $X^{s+\alpha}$ provided $u$ is in the class $X^{s+1}$, while $e$ is in the class $X^s$.

• **Step** #3 (*Higher regularity control*). The necessary bounds sought in (2.8) and (2.10) may require a restricted set of initial configurations depending on finite critical threshold assumed in (2.7). Whether these thresholds $\eta_\pm$ are restricted or not, the corresponding bounds will be derived solely on the basis of the mass equation for $\rho$, and the fact that $e = u_x + \mathcal{L}(\rho)$ satisfies the transport equation (2.5). This argument can be iterated to higher derivatives as follows. Note that if a quantity $Q$ is transported, $Q_t + uQ_x = 0$, then the *same* transport equation governs $Q_x/\rho$

$$\left(\frac{Q_x}{\rho}\right)_t + u\left(\frac{Q_x}{\rho}\right)_x = 0. \tag{2.11}$$

Let us apply this argument to $Q = e/\rho$: then if $|(e/\rho)_x|/\rho$ is bounded at $t = 0$ it will remain bounded at later time. Unraveling the formulas, we obtain the point-wise bound

$$|e'(x,t)| \leqslant C(e_\pm, \rho_\pm)|\rho'(x,t)|. \tag{2.12}$$

This control bound will become a key tool in proving Theorem 1.3.

Now that $(e/\rho)_x/\rho$ is transported, we can apply the argument above repeatedly to obtain a hierarchy of pointwise bounds

$$|\, e^{(k)}(x,t)| \leqslant C|\rho^{(k)}(x,t)|, \qquad k = 0, 1, \ldots \tag{2.13}$$

It is therefore clear that such bounds would allow to apply the same control principle as stated above in extending our results into higher order Sobolev spaces. However, we will leave to pursue this direction to a future work.

## 3. Bounded density in one-dimensional equations in commutator form

In this section, we implement the above strategy for global regularity in the presence of commutator forcing, $\mathcal{T}_\phi$, depending on the properties of the mollifier $\phi$. We begin with a general discussion on the boundedness of the density sought in Step #1. Here, the bound (2.8) is driven by the diffusive character of the mass equation, which is revealed once we rewrite the mass equation of (1.9) in the form

$$\rho_t + u\rho_x = -e\rho + \rho\mathcal{L}(\rho). \tag{3.1}$$

In view of the uniform bound (2.9), we see that $e\rho \sim \rho^2$ behaves as a quadratic term. This implies

$$-\eta_+\rho^2 + \rho\mathcal{L}(\rho) \leqslant \rho_t + u\rho_x \leqslant -\eta_-\rho^2 + \rho\mathcal{L}(\rho). \tag{3.2}$$

We turn to check step #1 in the three cases of interest. Here and throughout $|\cdot|_p$, $1 \leqslant p \leqslant \infty$, denotes the $L^p$-norm.

### 3.1. *Bounded density with bounded mollifiers $\mathcal{L}_\phi$, $\phi \in L^\infty_\#$*

Consider the case of $\mathcal{L} = \mathcal{L}_\phi = \int \phi(|x-y|)(\rho(y) - \rho(x))\,\mathrm{d}y$ with $\phi \in L^\infty_\#$ which is assumed to have a positive mass $\int \phi(r)\,\mathrm{d}r > 0$. We emphasize that $\phi$ need not be positive. We verify the boundedness of $\rho$ using the straightforward bound

$$-I(\phi)\rho - |\phi|_\infty M_0 \leqslant \mathcal{L}_\phi(\rho) \leqslant -I(\phi)\rho + |\phi|_\infty M_0, \qquad I(\phi) := \int \phi(r)\,\mathrm{d}r > 0. \tag{3.3}$$

Inserted into (3.2) we find

$$-\big(\eta_+ + I(\phi)\big)\rho^2 - |\phi|_\infty M_0\rho \leqslant \rho_t + u\rho_x \leqslant -\big(\eta_- + I(\phi)\big)\rho^2 + |\phi|_\infty M_0\rho.$$

The inequality on the right shows that the density remains uniformly bounded from above for *any* $\eta_-$ satisfying $\eta_- > -I(\phi)$, that is, $\rho(\cdot,t) \leqslant \rho_+$ provided (2.7) holds for such $\eta_-$'s,

$$u_0'(x) + \phi * \rho_0(x) - I(\phi)\rho_0(x) \geqslant \eta_-\rho_0(x), \qquad \eta_- > -I(\phi). \tag{3.4}$$

The inequality of the left then shows that along characteristics, $\dot\rho \geqslant -c\rho$ with $c := |\phi|_\infty M_0 + (\eta_+ + I(\phi))\rho_+$, and hence the density is bounded away from vacuum by the lower-bound $\rho(t) \gtrsim e^{-ct}$.

### 3.2. *Bounded density with singular mollifiers $\mathcal{L}_\alpha$, $\alpha < 2$*

To bound the density from above, we consider the case of positive mollifiers which are singular in the sense that

$$\lim_{r\downarrow 0} rm_\phi(r) \uparrow \infty, \qquad m_\phi(r) := \min_{|z|\leqslant r} \phi(|z|). \tag{3.5}$$

In this case, we use the bound

$$\begin{aligned}\mathcal{L}(\rho)(x_+) &\leqslant \int_{|x_+-y|\leqslant r} \phi(|x_+ - y|)(\rho(y) - \rho_+)\,\mathrm{d}y \\ &\leqslant m_\phi(r) \int_{|x_+-y|\leqslant r} (\rho(y) - \rho_+)\,\mathrm{d}y \leqslant m_\phi(r) M_0 - 2rm_\phi(r)\rho_+.\end{aligned}$$

By assumption, for *any* $\eta_- \leqslant 0$ we can choose a small enough $r = r_+$ such that $2r_+m_\phi(r_+) = 1 - \eta_-$ and the bound on the right of (3.2) then implies that the maximal value of the density $\rho_+(t) = \rho(x_+(t), t)$ satisfies

$$\dot{\rho}_+ \leqslant -\eta_-\rho_+^2 - 2r_+m_\phi(r_+)\rho_+^2 + c_0\rho_+ \leqslant -\rho_+^2 + c_0\rho_+, \qquad c_0 = m_\phi(r_+)M_0$$

Thus, $\rho(\cdot, t)$ remains bounded from above. We conclude that for singular kernels satisfying (3.5) , the density remains upper-bounded *independent* of the lower threshold $\eta_-$. In particular, this applies to $\phi_\alpha(r) = r^{-(1+\alpha)}, \alpha < 2$.

We turn to the lower bound on the density away from vacuum. For positive $\phi$'s, whether singular or not, we have[3]

$$\mathcal{L}_\phi(\rho)(x_-) = \int_y \phi(|x_- - y|)(\rho(y) - \rho(x_-))\rho(y)\,\mathrm{d}y \geqslant 0.$$

Therefore, the inequality on the left of (3.2) implies that minima values of the density, $\rho_-(t) = \rho(x_-(t), t)$ at *any interior* point $x_-(t) = \arg\min_{|y|\leqslant R}\{\rho(y, t)\}$ with $|x_-| < R$, satisfy $\dot{\rho}_- \geqslant -\eta_+\rho_-^2$ and hence $\rho(\cdot, t) > 0$. In the particular case of the torus $\Omega = \mathbb{T}^1$, we conclude with a *uniform* lower bound away from vacuum

$$\rho(\cdot, t) \geqslant \rho_-(t) = \frac{(\rho_0)_-}{t\eta_+(\rho_0)_- + 1}, \qquad (\rho_0)_- = \min_{x\in\mathbb{T}^1} \rho_0(x) > 0. \tag{3.6}$$

### 3.3. *Bounded density with NS equations $\mathcal{L}_2 = \partial_{xx}$*

We use the regularization coming from the parabolic part of the mass equation which becomes evident when (1.13) is written in the form

$$\rho_t + u\rho_x + e\rho = \rho\rho_{xx}. \tag{3.7}$$

†This is a special case of the monotonicity condition (2.1) with $(f, g) = (\mathbf{1}, \rho)$ implies $\mathcal{L}(\rho)(x_-) \geqslant \mathcal{L}(\mathbf{1}(x_-))\rho_- = 0$.

Recalling $e \geqslant \eta_+\rho$ it implies $\dot{\rho}_- \geqslant -\eta_+\rho_-^2$, which in turn yields recovers the same lower-bound (3.6). Trying to pursue the same argument for an upper-bound of the density fails when using the right-hand side of (3.2). Instead, we note that the quantity $f := u + \rho_x$ is the primitive of $e$ and hence satisfies the transport equation $f_t + uf_x = 0$. This follows by direct computation of (1.9)

$$u_t + uu_x = -(\rho_t' + u\rho_x').$$

It follows that

$$|\rho'(\cdot, t)| \leqslant 2|u_0|_\infty + |\rho_0'|_\infty. \tag{3.8}$$

Given the uniform bound on $\rho'$ and since we already proved that $\rho > 0$, (3.8) ties the upper bound for $\rho$ as well. We note in passing that even though we can now express the density equation as a *pure* diffusion

$$\rho_t = \rho\rho_{xx} + F \tag{3.9}$$

with bounded forcing $F = -u\rho_x - \rho e \in L^\infty$, we can only reach the end-point Schauder estimate $\rho_{xx} \in BMO$ (see Schlag, 1996), which is not enough to secure a uniform bound of $\mathcal{L}(\rho) = \rho_{xx}$ necessary to get control over the slope $u_x$. We will provide additional details how to reach that bound, which is needed for the global existence of NS equations in Section 5 below.

## 4. Global existence: bounded mollifiers, $\mathcal{L} = \mathcal{L}_\phi$

With regard to Theorem 1.1, it is straightforward to verify Step #2 in the case of bounded mollifiers—in view of (3.3), the upper bound of $\rho$ implies that $\mathcal{L}(\rho)$ is uniformly bounded, $|\mathcal{L}_\phi(\rho) + I(\phi)\rho_+| \leqslant |\phi|_\infty M_0$, and hence $u_x = e - \mathcal{L}(\rho)$ is uniformly bounded. We conclude the global regularity for sub-critical initial data satisfying (3.4), namely, for a fixed $\epsilon > 0$ there holds

$$u_0'(x) + \phi * \rho_0(x) \geqslant \epsilon\rho_0(x), \qquad \epsilon > 0.$$

In the particular case of $\mathbb{T}^1$, this requires the positivity of $u_0' + \phi * \rho_0$ stated in theorem 1.1. This recovers the same critical threshold of positive mollifiers (1.11).

## 5. Global existence: Navier–Stokes equations, $\mathcal{L} = \Delta$

In this section, we will prove Theorem 1.2. Recall that the boundedness of $\rho$ conveys to boundedness of $e = u_x + \rho_{xx}$, via

$$|e(x,t)| \leqslant \eta|\rho(x,t)|, \qquad \eta := \max\{-\eta_-, \eta_+\} \tag{5.1}$$

and that $\rho$ satisfies further *a priori* $C^1$-regularity (3.8). Note that these low-regularity *a priori* bounds hold classically under the assumptions $u \in H^2$, $\rho \in H^3$, which are the spaces for which Theorem 1.2 is stated, and these are the lowest $H^n$-regularity spaces of integer order that justify the above computations. We now proceed by establishing *a priori* estimates in these spaces.

First, let us quantify control over the high-order regularity of $e$.

LEMMA 5.1 For each $n = 0, 1, \ldots$ we have the following a priori estimate

$$\partial_t |e|^2_{H^n} \leqslant C(|e|^2_{H^n} + |u|^2_{H^{n+1}})(|u_x|_\infty + |e|_\infty). \tag{5.2}$$

*Proof.* For $n = 0$ the Lemma follows easily by testing the $e$ equation (2.4). For $n = 1, \ldots$, let us differentiate (2.4) $n$ times and test with $e^{(n)}$. We obtain (dropping the integral signs)

$$\partial_t |e^{(n)}|_2^2 \lesssim u\, e^{(n+1)}\, e^{(n)} + \sum_{k=1}^{n+1} u^{(k)}\, e^{(n+1-k)}\, e^{(n)}.$$

For the first term, we integrate by parts to obtain trivially $|u\, e^{(n+1)}\, e^{(n)}| \lesssim |u_x|_\infty |e|^2_{H^n}$. For each of the remaining terms on the right, $k = 1, \ldots, n+1$, we apply Gagliardo–Nirenberg inequalities, $|\partial^i f|_{\frac{2n}{i}} \leqslant |f|_\infty^{1-\frac{i}{n}} |f|_{H^n}^{\frac{i}{n}}$, $1 \leqslant i \leqslant n$, obtaining

$$\begin{aligned} |u^{(k)}\, e^{(n+1-k)}\, e^{(n)}| &\leqslant |e^{(n)}|_2 |u_x^{(k-1)}|_{\frac{2n}{k-1}} |e^{(n+1-k)}|_{\frac{2n}{n+1-k}} \\ &\leqslant |e^{(n)}|_2 |u_x|_\infty^{1-\frac{k-1}{n}} |u_x|_{H^n}^{\frac{k-1}{n}} |e|_\infty^{1-\frac{n+1-k}{n}} |e|_{H^n}^{\frac{n+1-k}{n}} \\ &\leqslant |e^{(n)}|_2^{\frac{2n+1-k}{n}} |u_x|_{H^n}^{\frac{k-1}{n}} |u_x|_\infty^{1-\frac{k-1}{n}} |e|_\infty^{1-\frac{n+1-k}{n}} \end{aligned}$$

and by Young's inequality $|u^{(k)}\, e^{(n+1-k)}\, e^{(n)}| \leqslant (|e|^2_{H^n} + |u|^2_{H^{n+1}})(|u_x|_\infty + |e|_\infty)$ which completed the proof. □

We now proceed establishing bounds on $u_x$ and $u_{xx}$ in a sequence of increasing norms, which eventually will close the estimates together with Lemma 5.1. Recall

$$u_t + uu' = \rho u'' + 2\rho' u'. \tag{5.3}$$

Testing with $u$ and using (3.8) we obtain

$$\frac{1}{2}\partial_t |u|_2^2 = -\int \rho |u'|^2 + \int \rho' u u' \leqslant -\frac{1}{2}\int \rho |u'|^2 + \frac{1}{2} C_1^2 |u|_2^2, \qquad C_1 = 2|u_0|_\infty + |\rho_0'|_\infty.$$

This proves the natural energy bound $u \in L^\infty_t L^2_x \cap L^2_t H^1_x$. Next, we test with $-u''$ to obtain (dropping the integrals)

$$\frac{1}{2}\partial_t |u'|_2^2 = |uu'u''| - \rho |u''|^2 + 2|\rho' u' u''| \lesssim -\frac{1}{2}\rho |u''|^2 + |u|^2 |u'|^2 + |\rho'|^2 |u'|^2.$$

Using uniform bound on $u$ and (3.8),

$$\frac{1}{2}\partial_t |u'|_2^2 \leqslant -\frac{1}{2}\rho_- |u''|_2^2 + C_2 |u'|_2^2, \qquad \rho_- = \min \rho(\cdot, t) > 0,$$

which implies $u \in L^\infty_t H^1_x \cap L^2_t H^2_x$. In particular, this implies $|u_x(\cdot, t)|_\infty \in L^1$, and hence the estimates on $H^1$-norm of $e$ from Lemma 5.1 closes with an integrable multiplier on the right hand side of (5.2).

It remains to establish a further similar bound on $|u''|_2$ to close the estimate on the grand quantity $|u''|_2^2 + |e'|_2^2 \sim |u''|_2^2 + |\rho|_{H^3}^2$. So, we differentiate (5.3) twice and test with $u''$:

$$\begin{aligned}\frac{1}{2}\partial_t|u''|_2^2 + \frac{5}{2}u'u''u'' &= -(\rho u'')'u''' + 2\rho'''u'u'' + 4\rho''u''u'' + 2\rho'u'''u'' \\ &= -\rho|u'''|^2 + \rho'u'''u'' + 2\rho'''u'u'' + 4\rho''u''u''.\end{aligned}$$

So,

$$\begin{aligned}\frac{1}{2}\partial_t|u''|_2^2 &\lesssim |u_x|_\infty|u''|_2^2 - \frac{1}{2}\rho_-|u'''|_2^2 + |\rho'|_\infty|u''|_2^2 + |u_x|_\infty|\rho'''|_2|u''|_2 + 4|e-u_x|_\infty|u''|_2^2 \\ &\lesssim -\frac{1}{2}\rho_-|u'''|_2^2 + |u_x|_\infty(|e'|_2^2 + |u''|_2^2).\end{aligned}$$

Given the established integrability of $|u_x|_\infty$ and Lemma 5.1 we have proved boundedness in $H^2$ for $u$, and $H^3$ for $\rho$.

## 6. Global existence: singular mollifiers $\mathcal{L} = \mathcal{L}_\alpha, 1 \leqslant \alpha < 2$

In this section, we prove global regularity result for the equation with fractional $\mathcal{L} = -\Lambda_\alpha$ in space of data $H^3 \times H^{2+\alpha}$. The case $\alpha = 1$ is critical similar to the classical fractional Burgers equation, (Kiselev *et al.*, 2008; Caffarelli & Vasseur, 2010; Constantin & Vicol, 2012) but with additional nonlinearity in the dissipation term. We will leave the subcritical case $\alpha > 1$ as an easy consequence of the proof presented here for the case $\alpha = 1$. Note that with the initial datum $(u_0, \rho_0)$ in $H^3$ we can avoid making assumptions on $e_0$ as $e_0 \in H^2 \subset C^{1+\gamma}$ for any $\gamma < 1/2$ by the Sobolev embedding. With this we recall *a priori* uniform bounds from the previous section,

$$\sup_{0<t<T}|e|_\infty < \infty, \quad 0 < \rho_- \leqslant \rho(x,t) \leqslant \rho_+, \quad u_- \leqslant u(x,t) \leqslant u_+ \tag{6.1}$$

on any finite time interval of existence. The lower bound on the density is the main reason why we resort to the periodic domain. In the open space such bound is only known to hold on any finite interval, lacking a uniform parabolicity to the system.

Moreover, for a solution in $H^3$ the transport equation (2.11) for $Q = e/\rho$ can be solved classically along characteristics of $u$ which results in the bound (2.12) which we quote for convenience

$$|e_x(x,t)| \leqslant C|\rho_x(x,t)|, \text{ for all } (x,t) \in [0,T) \times \mathbb{T}. \tag{6.2}$$

The proof will consist of four steps. First, we establish the local existence in $H^3$ by obtaining rough *a priori* bounds without exploiting dissipation term. This allows to perform classical desingularization of the kernel as an approximate scheme to obtain local solutions. Second, we establish uniform control over first order quantities $|\rho_x|_\infty$, $|u_x|_\infty$ over the interval of regularity. The strategy here resembles the treatment of the critical SQG by Constantin & Vicol (2012), but with additional technicalities related to the nonlinear nature of the dissipation term. We then invoke the results of Schwab & Silvestre (2012) to obtain instantaneous $C^\gamma$-regularization and use it to have an easier control on the oscillations in the midrange of scales of the nonlinearity. Third, we establish uniform control over $H^2$ norm of solutions by proving an analogue of the Beale–Kato–Majda estimates. With the $H^2$ and $W^{1,\infty}$ bounds, we finally conclude by a proving a uniform control of the penultimate $H^3$-norm of the solution on the entire interval of existence.

It will be useful to introduce the following notation. For three functions $f, g, h$ of $x, z$ we denote

$$\Phi(f,g,h) := \frac{1}{2}\iint \frac{f(x,z)g(x,z)h(x,z)}{|z|^2}\,\mathrm{d}z\,\mathrm{d}x.$$

Moreover, for a cutoff function $\varphi$ and parameter $r > 0$ we denote

$$\begin{aligned}\Phi_{<r}(f,g,h) &= \frac{1}{2}\iint \frac{f(x,z)g(x,z)h(x,z)}{|z|^2}\varphi(z/r)\,\mathrm{d}z\,\mathrm{d}x\\ \Phi_{>r}(f,g,h) &= \frac{1}{2}\iint \frac{f(x,z)g(x,z)h(x,z)}{|z|^2}(1-\varphi(z/r))\,\mathrm{d}z\,\mathrm{d}x.\end{aligned}$$

In the sequel we will also use the following notation $\delta_z f(x) = f(x+z) - f(x)$, and the expansion

$$\delta_z f(x) = f'(x)z + z^2\int_0^1 (1-\theta)f''(x+\theta z)\,\mathrm{d}\theta. \tag{6.3}$$

### 6.1. *Local well-posedness in $H^3$: a priori estimates without the use of dissipation*

The purpose of this section is to obtain a priori estimates in $H^3$ which do not rely on the dissipation term. Namely, we will obtain the classical Riccati equation for the quantity $Y = |u|_{H^3} + |\rho|_{H^3} \sim |u|_{H^3} + |e|_{H^2} + |\rho|_2$:

$$Y_t \leqslant CY^2,$$

which is independent of desingularization of the kernel $K_\delta = \frac{1}{(|z|^2+\delta^2)^{\frac{n+1}{2}}}$. This allows to conclude local existence via the classical approximation methods.

Let us write the equation for $u'''$:

$$u'''_t + uu'''_x + 4u'u''' + 3u''u'' = \mathcal{T}(\rho''',u) + 3\mathcal{T}(\rho'',u') + 3\mathcal{T}(\rho',u'') + \mathcal{T}(\rho,u'''). \tag{6.4}$$

Testing with $u'''$ we obtain (we suppress integral signs and note that $\int u''u''u''' = 0$)

$$\partial_t|u'''|_2^2 = -7u'(u''')^2 + 2\mathcal{T}(\rho''',u)u''' + 6\mathcal{T}(\rho'',u')u''' + 6\mathcal{T}(\rho',u''))u''' + 2\mathcal{T}(\rho,u''')u'''. \tag{6.5}$$

We will now perform several estimates with the purpose of extracting term $|u'''|_2^2$ on the right hand side, times a lower order term in $u$ and possibly a top order term in $\rho$ which we will address subsequently. First, we have trivially

$$|u'(u''')^2| \leqslant |u'|_\infty |u'''|_2^2. \tag{6.6}$$

Let us estimate the dissipative term first:

$$\int \mathcal{T}(\rho, u''')u''' dx = \iint \rho(y)u'''(x)(u'''(y) - u'''(x))\frac{\mathrm{d}y\ \mathrm{d}x}{|x-y|^2}.$$

Switching $x$ and $y$ and adding cross-terms $\rho(x)u'''(x)$ we obtain

$$\begin{aligned}\int \mathcal{T}(\rho, u''')u''' dx = &-\frac{1}{2}\iint \rho(x)(u'''(y) - u'''(x))^2\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^2}\\ &+\frac{1}{2}\iint u'''(x)(\rho(y)-\rho(x))(u'''(y) - u'''(x))\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^2}.\end{aligned}$$

The first term is clearly negative. We note in passing that it is bounded below by

$$\iint \rho(x)(u'''(y) - u'''(x))^2\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^2} \geqslant \rho_-|u'''|^2_{H^{1/2}}.$$

While it is undoubtedly a crucial piece of information, it does depend on the fact that the kernel is singular. As we indicated earlier, however, we seek estimates that are independent of singularity. So, at this point we will simply dismiss the dissipation term. As to the second term, we rewrite it as

$$\iint u'''(x)(\rho(y)-\rho(x))(u'''(y) - u'''(x))\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^2} = \Phi(u''', \delta_z\rho, \delta_z u''').$$

We estimate the large-scale part of the integral using integrability of $|z|^{-2}$ at infinity as follows

$$\Phi_{>1}(u''', \delta_z\rho, \delta_z u''') \leqslant |\rho|_\infty |u'''|_2^2. \tag{6.7}$$

As to the small scale, we use the expansion (6.3) on $\rho$. We have

$$\begin{aligned}\Phi_{<1}(u''', \delta_z\rho, \delta_z u''') = &\iint \varphi(z)u'''(x)\rho'(x)\delta_z u'''(x)\frac{\mathrm{d}z\,\mathrm{d}x}{z}\\ &+\int_0^1(1-\theta)\iint \varphi(z)u'''(x)\rho''(x+\theta z)\delta_z u'''(x)\,\mathrm{d}z\,\mathrm{d}x\,d\theta.\end{aligned}$$

Writing the first integral in the principal value sense results in the cancellation

$$\iint \varphi(z)u'''(x)\rho'(x)u'''(x)\frac{\mathrm{d}z\,\mathrm{d}x}{z} = 0,$$

while

$$\iint \varphi(z)u'''(x)\rho'(x)u'''(x+z)\frac{\mathrm{d}z\,\mathrm{d}x}{z} = \int u'''(x)\rho'(x)H_\varphi(u''')(x)\,\mathrm{d}x,$$

where $H_\phi$ is the truncated Hilbert transform given by convolution with the kernel $\frac{\varphi(z)}{z}$. It is a bounded operator on $L^2$. We thus have the estimate

$$\left|\int u'''(x)\rho'(x)H_\varphi(u''')(x)\,\mathrm{d}x\right| \leqslant |\rho'|_\infty |u'''|_2^2.$$

Putting the estimates together, we arrive at the bound

$$\int \mathcal{T}(\rho,u''')u'''\,\mathrm{d}x \leqslant C|u'''|_2^2(|\rho|_\infty + |\rho'|_\infty). \tag{6.8}$$

We now proceed with the remaining three terms in (6.5) in a similar fashion. We have

$$\begin{aligned}
\mathcal{T}(\rho''',u)u''' &= \Phi(\rho'''(\cdot+z),u''',\delta_z u) = \Phi_{>1}(\rho'''(\cdot+z),u''',\delta_z u) + \Phi_{<1}(\rho'''(\cdot+z),u''',\delta_z u)\\
&\leqslant |u'''|_2|\rho'''|_2|u|_\infty + \int H_\varphi(\rho''')(x)u'''(x)u'(x)\,\mathrm{d}x\\
&\quad + \int_0^1 (1-\theta)\iint \rho'''(x+z)u'''(x)u''(x+\theta z)\varphi(z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}\theta\\
&\leqslant |u'''|_2|\rho'''|_2(|u|_\infty + |u'|_\infty + |u''|_\infty).\\
\mathcal{T}(\rho'',u')u''' &= \Phi(\rho''(\cdot+z),u''',\delta_z u') = \Phi_{>1}(\rho''(\cdot+z),u''',\delta_z u') + \int H_\varphi(\rho'')(x)u'''(x)u''(x)\,\mathrm{d}x\\
&\quad + \int_0^1 (1-\theta)\iint \rho''(x+z)u'''(x)u'''(x+\theta z)\varphi(z)\;\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}\theta\\
&\leqslant |u'''|_2|\rho''|_2|u'|_\infty + |u'''|_2|\rho''|_2|u''|_\infty + |u'''|_2^2|\rho''|_\infty\\
&= |u'''|_2^2|\rho''|_\infty + |u'''|_2|\rho''|_2(|u'|_\infty + |u''|_\infty).
\end{aligned}$$

And the last term requires more preparation,

$$\begin{aligned}
\mathcal{T}(\rho',u'')u''' &= \iint \rho'(y)u'''(x)(u''(y)-u''(x))\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^2}\\
&= \frac{1}{2}\iint (\rho'(y)u'''(x)-\rho'(x)u'''(y))(u''(y)-u''(x))\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^2}\\
&= \frac{1}{2}\iint (\rho'(y)-\rho'(x))u'''(x)(u''(y)-u''(x))\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^2}\\
&\quad + \frac{1}{2}\iint \rho'(x)(u'''(x)-u'''(y))(u''(y)-u''(x))\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^2}\\
&= \frac{1}{2}\Phi(\delta_z\rho',u''',\delta_z u'') - \frac{1}{2}\Phi(\rho',\delta_z u''',\delta_z u'')\\
&= \frac{1}{2}\Phi_{>1}(\delta_z\rho',u''',\delta_z u'') + \frac{1}{2}\iint \varphi(z)\rho''(x)u'''(x)\delta_z u''(x)\frac{\mathrm{d}z}{z}\;\mathrm{d}x\\
&\quad + \frac{1}{2}\int_0^1 (1-\theta)\iint \varphi(z)\rho'''(x+\theta z)u'''(x)\delta_z u''(x)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}\theta
\end{aligned}$$

$$
- \frac{1}{4} \iint \rho'(x)((\delta_z u''(x))^2)' \frac{\mathrm{d}z}{|z|^2}\ \mathrm{d}x
$$

$$
\leqslant |\rho'|_\infty |u'''|_2 |u''|_2 + |\rho''|_\infty |u'''|_2 |u''|_2 + |\rho'''|_2 |u'''|_2 |u''|_\infty + \frac{1}{4} \iint \rho''(x)(\delta_z u''(x))^2 \frac{\mathrm{d}z}{|z|^2}\ \mathrm{d}x.
$$

The latter integral is bounded by $|\rho''|_\infty |u''|^2_{\dot{H}^{1/2}} \leqslant |\rho''|_\infty |u'''|_2 |u''|_\infty$. Putting the obtained estimates together we obtain

$$
\begin{aligned}
\partial_t |u'''|_2^2 \leqslant C|u'''|_2^2(|u'|_\infty + |\rho|_\infty + |\rho'|_\infty + |\rho''|_\infty) \\
+ |u'''|_2(|\rho''|_2 + |\rho'''|_2 + |\rho''|_\infty)(|u|_\infty + |u'|_\infty + |u''|_\infty) \\
+ |u'''|_2 |u''|_2(|\rho'|_\infty + |\rho''|_\infty).
\end{aligned} \tag{6.9}
$$

Finally, by Sobolev embedding, $|u''|_2 + |u|_\infty + |u'|_\infty + |u''|_\infty \leqslant C(|u'''|_2 + |u|_2)$, and $|\rho|_\infty + |\rho'|_\infty + |\rho''|_\infty + |\rho''|_2 \leqslant C(|\rho'''|_2 + |\rho|_2)$ which results in the bound

$$
\partial_t |u'''|_2^2 \lesssim (|u'''|_2 + |u|_2)^2(|\rho'''|_2 + |\rho|_2) + (|u'''|_2 + |u|_2)^3. \tag{6.10}
$$

To control the energy $|u|_2$ we avoid using the natural balance relation (1.8). Instead we test (1.1) directly with $u$. Performing much the same estimates as above we obtain, for example,

$$
\partial_t |u|_2^2 \leqslant |u|_\infty |u|_2 |\rho|_2 + |\rho'|_2 |u'|_2 |u|_\infty.
$$

Putting this together with (6.10) we obtain the Riccati equation for the $H^3$-norm:

$$
\partial_t |u|_{H^3} \leqslant |u|_{H^3} |\rho|_{H^3} + |u|^2_{H^3}. \tag{6.11}
$$

In order to close the estimates we now have to find a similar bound on the $H^3$-norm of $\rho$. This cannot be done directly by manipulating with the density transport equation. Instead we will make use of the transport of the first order quantity $e$, in terms of which we will provide the final estimates. Let us note the inequality

$$
|\rho|_{H^3} \leqslant |u|_{H^3} + |e|_{H^2} + |\rho|_2.
$$

Thus,

$$
\partial_t |u|_{H^3} \leqslant |u|_{H^3}(|e|_{H^2} + |\rho|_2) + |u|^2_{H^3}. \tag{6.12}
$$

From Lemma 5.1, we have the bound on $|e|_{H^2}$:

$$
\partial_t |e|_{H^2} \leqslant C(|e|_{H^2} + |u|_{H^3})^2. \tag{6.13}
$$

And the similar bound holds for $|\rho|_2$. We have obtained the classical Riccati equation for the quantity $Y = |u|_{H^3} + |e|_{H^2} + |\rho|_2$:

$$
Y_t \leqslant CY^2.
$$

Note that $Y \sim |u|_{H^3} + |\rho|_{H^3}$, hence we have proved necessary a priori bound for the local well-posedness in $H^3$.

### 6.2. *Control over $|u_x|_\infty$ and $|\rho_x|_\infty$ on intervals of regularity*

Suppose that we have a classical solution $(u, \rho) \in C([0, T); H^3)$ as proved to exist in the previous section. We now seek to establish a uniform bound on $|u_x|_\infty$ and $|\rho_x|_\infty$ on the entire interval $[0, T)$. First, let us recall that we have already established *a priori* uniform bounds of $e, \rho$ and $u$ in terms of the finite initial quantities $e_\pm$, $\rho_\pm$ and $u_\pm$, consult (6.1). Next, as we noted the density $\rho$ satisfies a parabolic form of the density equation:

$$\rho_t + u\rho_x + e\rho = \rho\mathcal{L}(\rho). \tag{6.14}$$

Similarly, one can write the equation for the momentum $m = \rho u$:

$$m_t + um_x + em = \rho\mathcal{L}(m). \tag{6.15}$$

Note that in both cases the drift $u$ and the forcing $e\rho$ or $em$ are bounded *a priori*. Moreover, the diffusion operator has kernel

$$K(x, h, t) = \rho(x)\frac{1}{|h|^2}$$

which satisfies all the assumptions of Schwab & Silvestre (2012). A direct application of Schwab & Silvestre (2012) tells us that there exists an $\gamma > 0$ such that

$$\begin{aligned}
|\rho|_{C^\gamma(\mathbb{T}\times[T/2,T))} &\leqslant C(|\rho|_{L^\infty(0,T)} + |\rho e|_{L^\infty(0,T)}) \\
|m|_{C^\gamma(\mathbb{T}\times[T/2,T))} &\leqslant C(|m|_{L^\infty(0,T)} + |me|_{L^\infty(0,T)}) \\
|u|_{C^\gamma(\mathbb{T}\times[T/2,T))} &\leqslant C(|u|_{L^\infty(0,T)}, |\rho|_{L^\infty(0,T)}),
\end{aligned} \tag{6.16}$$

where the latter follows from the first two since $\rho$ is bounded below. Of course, since $u, \rho$ are in $H^3$ on $[0, T)$ this implies $C^\gamma$-bound on the entire interval of regularity, however we need the bound to be independent of $H^3$, which may blow up, in the second half of it. It is also interesting to note that the original equation for $u$ has a kernel $K(x, h, t) = \rho(x + h)\frac{1}{|h|^2}$ not even with respect to $h$, so no known results on regularization are directly applicable to the $u$-equation.

REMARK 6.1 In regard to higher-order regularization via Schauder, we make the following observation. For $Q = e/\rho$, we recall that $Q_x$ was shown to be under control (note that this still doesn't imply that either $e_x$ or $\rho_x$ are under control). Hence, trivially, $|Q|_{C^\gamma}$ remains bounded at all times. Denote

$$\delta_h Q(x) = \frac{Q(x + h) - Q(x)}{|h|^\gamma}$$

and note

$$\delta_h Q(x) = \frac{\delta_h e(x)}{\rho(x + h)} + \frac{e(x)\delta_h\rho(x)}{\rho(x + h)\rho(x)}.$$

Since $\rho$ is $C^\gamma$ and bounded away from zero this implies that $e \in C^\gamma$ with $|e(t)|_{C^\gamma} \leqslant C/t^\gamma$. With this in mind, we now have the momentum equation in the form

$$m_t + b(x)m_x + a(x)\Lambda m = F,$$

where the drift $b$, the coefficient function $a$ and the source $F$ are all in $C^\gamma$. As of this writing there has been no known Schauder-type bounds proved for an equation in such generality despite many recent developments in that cover partial cases (see Chang-Lara & Kriventsov, 2015; Dong & Zhang, 2015; Schwab & Silvestre, 2012; Jin & Xiong, 2015). The question presents an independent interest and we will address it in subsequent work.

Let us now establish control over $\rho'$. We write

$$\partial_t \rho' + u\rho'' + u'\rho' + e'\rho + e\rho' = -\rho'\Lambda\rho - \rho\Lambda\rho'.$$

Using again $u' = e + \Lambda\rho$ we rewrite

$$\partial_t \rho' + u\rho'' + e'\rho + 2e\rho' = -2\rho'\Lambda\rho - \rho\Lambda\rho'.$$

Let us evaluate it at the maximum of $\rho'$ and multiply by $\rho'$ again (we use the classical Rademacher theorem here to justify the time derivative):

$$\partial_t|\rho'|^2 + e'\rho\rho' + 2e|\rho'|^2 = -2|\rho'|^2\Lambda\rho - \rho\rho'\Lambda\rho'. \tag{6.17}$$

In view of (6.1) and (6.2) we can bound

$$|e'\rho\rho' + 2e|\rho'|^2| \leqslant C|\rho'|^2.$$

Next, using the nonlinear bounds from Constantin & Vicol (2012), we have

$$\rho\rho'\Lambda\rho' \geqslant \frac{1}{4}\rho_- D\rho'(x) + c\frac{\rho_-}{\rho_+}|\rho'|_\infty^3 \geqslant c_1 D\rho'(x) + c_2|\rho'|_\infty^3, \tag{6.18}$$

where

$$D\rho'(x) = \int_{\mathbb{R}} \frac{|\rho'(x) - \rho'(x+z)|^2}{|z|^2}\,\mathrm{d}z.$$

Using smooth decompositions of the underlying $\mathbb{R}$ in all of the below we have

$$\Lambda\rho(x) = H\rho' = \int_{|z|<r} \frac{\rho'(x+z) - \rho'(x)}{z}\,\mathrm{d}z - \int_{r<|z|<2\pi} \frac{\rho(x+z)-\rho(x)}{|z|^2}\,\mathrm{d}z$$
$$- \int_{2\pi<|z|} \frac{\rho(x+z)-\rho(x)}{|z|^2}\,\mathrm{d}z.$$

The latter is clearly bounded by a constant $c_3$ depending only on $\rho_-$, which in (6.17) results simply in the bound $c_3|\rho'|_\infty^2$. The first is bounded, via Hölder, by

$$|\rho'|_\infty^2 \sqrt{r} D^{1/2}\rho'(x) \leqslant \frac{1}{2}c_1 D\rho'(x) + c_4 r|\rho'|_\infty^4.$$

Note that this term gets absorbed by the dissipation (6.18) entirely if

$$r = \frac{c_2}{4c_4|\rho'|_\infty}.$$

The integral in the middle is bounded by, using $C^\gamma$-regularity,

$$|\rho'|_\infty^2|\rho|_{C^\gamma}/r^{1-\gamma} = c_5|\rho'|_\infty^{3-\gamma} \leqslant c_6 + \frac{c_2}{4}|\rho'|_\infty^3,$$

where the cubic term again is absorbed by the dissipation. Putting the estimates together we obtain

$$\partial_t|\rho'|^2 \leqslant c_6 + c_3|\rho'|^2 - c_7 D\rho'(x), \tag{6.19}$$

which establishes the claimed control of $\rho'$. We intentionally keep the dissipation term as it still will be used on the next step to absorb other terms.

Now we can do the same for the momentum derivative $m_x$. Clearly it is sufficient to finish the proof for $u_x$ as well. Note that the equation for momentum is similar, so we will skip details that are similar. We have

$$\partial_t m' + um'' + u'm' + e'm + em' = -\rho'\Lambda m - \rho\Lambda m'.$$

Evaluating at maximum, multiplying by $m'$, and using bounds on $e, e'$ we have

$$\partial_t|m'|^2 \leqslant c_8(|m'|_\infty^2 + |\rho'|_\infty|m'|_\infty) + |m'|^2|\Lambda\rho| + |\rho'||m'||\Lambda m| - c_9 Dm'(x) - c_{10}|m'|_\infty^3. \tag{6.20}$$

As to $|\rho'||m'||\Lambda m|$ we proceed as before, losing $\rho'$ in view of already established control over it. We obtain the bound simply by taking $r = 1$:

$$c_{11}|m'| + |m'|D^{1/2}m'(x) \leqslant c_{11}|m'| + c_{12}|m'|^2 + \frac{c_9}{4}Dm'(x)$$

with the latter being absorbed again in the dissipation. As to the term $|m'|^2|\Lambda\rho|$ we still proceed as before, however in the mid-range integral $r < |z| < 2\pi$ we use the full force of the obtained bound on $\rho'$. This results in logarithmic optimization bound

$$|m'|^2|\Lambda\rho| \leqslant c_{13}|m'|^2(1 + \ln r + \sqrt{r}D^{1/2}\rho'(x)).$$

Ignoring the trivial quadratic term $|m'|^2$, we have

$$c_{13}|m'|^2\ln r + c_{13}|m'|^2\sqrt{r}D^{1/2}\rho'(x) \leqslant c_{13}|m'|^2\ln r + c_{14}|m'|^4 r + \frac{c_7}{2}D\rho'(x).$$

Notice that the latter will be absorbed by the dissipation term in (6.19) when we add the two equations together. Choosing

$$r = \frac{c_{10}}{2c_{14}|m'|},$$

we obtain for the $\ln r$ and $r$-terms above the bound

$$c_{15}|m'|^2 \ln |m'| + \frac{c_{10}}{2}|m'|^3$$

with the latter being absorbed into the cubic term in (6.20). Altogether we have

$$\partial_t |m'|^2 \leqslant c_{16}|m'|^2(1 + \ln_+ |m'|) + \frac{c_7}{2} D\rho'(x). \tag{6.21}$$

We now have to add the two equations (6.21) and (6.19) together to absorb the residual $D\rho'$-term and obtain the final bound

$$\partial_t(|m'|^2 + |\rho'|^2) \leqslant c_{17}(|m'|^2 + |\rho'|^2)(1 + \ln_+(|m'|^2 + |\rho'|^2)). \tag{6.22}$$

This implies double-exponential, but finite, bound on the given interval. This also finishes the proof.

### 6.3. *Control over $H^2$ via $W^{1,\infty}$*

In this section, we will establish an estimate on the $H^2$-norm of the solution

$$X = |u''|_2^2 + |\rho''|_2^2 \sim |u''|_2^2 + |e'|_2^2$$

in terms of $|u_x|_\infty$ is a manner similar to the Beale–Kato–Majda criterion. Namely, we will prove

$$X' \leqslant C(1 + |u'|_\infty) X(1 + \log_+ X). \tag{6.23}$$

Given the result of the previous section, this establishes uniform bound in $H^2$ on the interval of existence $[0, T)$ of an $H^3$-solution. The equation for $u''$ reads

$$u''_t + uu''_x + 3u'u'' = \mathcal{T}(\rho'', u) + 2\mathcal{T}(\rho', u') + \mathcal{T}(\rho, u'').$$

Testing with $u''$ the local terms , after integration by parts , become bounded by $X|u'|_\infty$ trivially. We now look into key estimates for the right-hand side. We will start with what proved to be the most involved term in the previous section. We skip the standard symmetrization and addition of cross-product terms in the calculations below and typically display the final representations. We have

$$\iint \mathcal{T}(\rho', u')u'' \,\mathrm{d}x\,\mathrm{d}y = \Phi(\delta_z\rho', \delta_z u', u'') + \Phi(\rho', \delta_z u', \delta_z u''). \tag{6.24}$$

For the second term, we have $\delta_z u' \delta_z u'' = \frac{1}{2}((\delta_z u')^2)_x$. So, switching the derivative onto $\rho'$ we obtain

$$\Phi(\rho', \delta_z u', \delta_z u'') = -\frac{1}{2}\Phi(\rho'', \delta_z u', \delta_z u').$$

Now, we bound small and large scale parts as follows

$$|\Phi_{>r}(\rho'', \delta_z u', \delta_z u')| \leqslant \frac{1}{r}|\rho''|_2 |u'|_4^2$$

and

$$|\Phi_{<r}(\rho'', \delta_z u', \delta_z u')| \leqslant \sqrt{r}|\rho''|_2 |u'|^2_{W^{3/4,4}},$$

where in the latter we used the Hölder and Gagliardo–Sobolevskii definition of $W^{3/4,4}$ space. Optimizing over $r$ we obtain

$$|\Phi(\rho', \delta_z u', \delta_z u'')| \leqslant |\rho''|_2 |u'|_4^{2/3} |u'|^{4/3}_{W^{3/4,4}}$$

and by Gagliardo–Nirenberg,

$$|u'|_{W^{3/4,4}} \leqslant |u''|^{1/2}_{H^{1/2}} |u'|^{1/2}_\infty,$$

and interpolation we obtain

$$|\Phi(\rho', \delta_z u', \delta_z u'')| \leqslant |\rho''|_2 |u'|_4^{2/3} |u'|_\infty^{2/3} |u''|^{2/3}_{H^{1/2}} \leqslant \frac{1}{\varepsilon}|\rho''|_2^{3/2} |u'|_4 |u'|_\infty + \varepsilon |u''|^2_{H^{1/2}}.$$

With $\varepsilon < \rho_-/2$ the last term is absorbed by the dissipation. Finally, by Gagliardo–Nirenberg we have

$$|u'|_4 \leqslant |u''|_2^{1/2} |u|_\infty^{1/2}. \tag{6.25}$$

Recalling that $|u|_{W^{1,\infty}}$ is under control, we finally obtain

$$|\Phi(\rho', \delta_z u', \delta_z u'')| \leqslant C|\rho''|_2^{3/2} |u''|_2^{1/2} |u'|_\infty + \varepsilon |u''|^2_{H^{1/2}} \leqslant CX + \varepsilon |u''|_{H^{1/2}}.$$

For the other term $\Phi(\delta_z \rho', \delta_z u', u'')$ the splitting is necessary but optimization is not. We have, in view of (6.25),

$$\Phi_{>1}(\delta_z \rho', \delta_z u', u'') \leqslant |\rho'|_4 |u'|_4 |u''|_2 \lesssim |\rho''|_2^{1/2} |u''|_2^{3/2} \leqslant X.$$

As to $\Phi_{<1}$, we write $\delta_z u'(x) = \delta_z u'(x) - z u''(x) + z u''(x)$, and note that $|\delta_z u'(x) - z u''(x)| \leqslant |z|^{3/2} D^{1/2} u''(x)$. So, we have

$$\begin{aligned} |\Phi_{<1}(\delta_z \rho', \delta_z u', u'')| &\leqslant \left| \int H_\varphi \rho'(x) |u''(x)|^2 dx \right| + |\rho'|_\infty |Du''|_2 |u''|_2 \\ &\leqslant |H_\varphi \rho'|_\infty |u''|_2^2 + \frac{1}{\varepsilon} |\rho'|_\infty^2 |u''|_2^2 + \varepsilon |u''|^2_{H^{1/2}}. \end{aligned}$$

Note that since $H_\varphi$ is a bounded Fourier multiplier, by the log-Sobolev inequality, we have

$$|H_\varphi \rho'|_\infty \leqslant |\rho'|_\infty (1 + \log_+ |\rho''|_2) \leqslant |\rho'|_\infty (1 + \log_+ X).$$

So,

$$|\Phi_{<1}(\delta_z\rho', \delta_z u', u'')| \leqslant C(1 + |\rho'|_\infty + |\rho'|^2_\infty + |u'|_\infty)X(1 + \log_+ X) + \varepsilon|u''|^2_{H^{1/2}}.$$

Taking into account the control over $W^{1,\infty}$ norms, we have proved the bound

$$|\mathcal{T}(\rho', u')u''| \leqslant CX(1 + \log_+ X) + \varepsilon|u''|_{H^{1/2}}.$$

Next, let us bound the dissipation term

$$\iint \mathcal{T}(\rho, u'')u'' \,\mathrm{d}y\,\mathrm{d}x = -\Phi(\rho, \delta_z u'', \delta_z u'') + \Phi(\delta_z\rho, \delta_z u'', u'').$$

Obviously,

$$\Phi(\rho, \delta_z u'', \delta_z u'') \geqslant \rho_-|u''|^2_{H^{1/2}}.$$

As to $\Phi(\delta_z\rho, \delta_z u'', u'')$ we have

$$\Phi_{>r}(\delta_z\rho, \delta_z u'', u'') \leqslant \frac{1}{r}|u''|^2_2$$

and

$$\Phi_{<r}(\delta_z\rho, \delta_z u'', u'') \leqslant |\rho'|_\infty \int_{\mathbb{R}} |u''(x)| \int_{|z|<2r} \left|\frac{\delta_z u''}{z}\right| \mathrm{d}z\,\mathrm{d}x \leqslant \sqrt{r}|\rho'|_\infty |u''|_2 |u''|_{H^{1/2}}.$$

Optimizing we obtain

$$\begin{aligned}\Phi(\delta_z\rho, \delta_z u'', u'') &\leqslant |\rho'|^{2/3}_\infty |u''|^{4/3}_2 |u''|^{2/3}_{H^{1/2}} \leqslant \varepsilon|u''|^2_{H^{1/2}} + \frac{1}{\varepsilon}|\rho'|_\infty |u''|^2_2\\ &\leqslant \varepsilon|u''|^2_{H^{1/2}} + CX(1 + \log_+ X),\end{aligned}$$

which closes the estimates with the help of dissipation.

It remains to estimate the last term. By switching $x$ and $y$ we obtain

$$\begin{aligned}\iint \mathcal{T}(\rho'', u)u'' \,dydx &= \iint \rho''(x)(u(x) - u(y))u''(y)\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^2}\\ &= \iint \rho''(x)(u(x) - u(y))(u''(y) - u''(x))\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^2}\\ &\quad + \iint \rho''(x)u''(x)(u(x) - u(y))\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^2}\\ &= \Phi(\rho'', \delta_z u, \delta_z u'') + \int \rho'' u'' \Lambda(u)\,\mathrm{d}x.\end{aligned} \tag{6.26}$$

Clearly, by the log-Sobolev inequality,

$$\left|\int \rho'' u'' \Lambda(u)\,\mathrm{d}x\right| \leqslant |\rho''|_2 |u''|_2 |\Lambda u|_\infty \lesssim |u'|_\infty X(1+\log_+ X).$$

For the F-term, we have

$$|\Phi_{>r}(\rho'', \delta_z u, \delta_z u'')| \leqslant \frac{1}{r}|\rho''|_2 |u''|_2,$$

while

$$|\Phi_{<r}(\rho'', \delta_z u, \delta_z u'')| \leqslant |u'|_\infty \int |\rho''(x)| \int_{|z|<2r} \frac{|\delta_z u''(x)|}{|z|}\,\mathrm{d}z\,\mathrm{d}x \leqslant |u'|_\infty \sqrt{r}|\rho''|_2 |u''|_{H^{1/2}}.$$

Optimizing, we get

$$|\Phi(\rho'', \delta_z u, \delta_z u'')| \leqslant |\rho''|_2 |u''|_2^{1/3} |u'|_\infty^{2/3} |u''|_{H^{1/2}}^{2/3} \leqslant \varepsilon |u''|^2_{H^{1/2}} + \frac{1}{\varepsilon}|\rho''|_2^{3/2} |u''|_2^{1/2} |u'|_\infty \leqslant CX.$$

We have proved that

$$\partial_t |u''|_2^2 \leqslant -\varepsilon |u''|^2_{H^{1/2}} + CX(1+\log_+ X).$$

As to quantity $e$, we apply Lemma 5.1 to obtain

$$\partial_t |e'|_2^2 \leqslant CX.$$

Putting the estimates together, (6.23) follows.

### 6.4. *Control over $H^3$ via $H^2$ and $W^{1,\infty}$*

For a given classical solution $(u,\rho) \in C([0,T); H^3)$ we have established uniform bounds on $|u_x, \rho_x|_\infty$ and $|u,\rho|_{H^2}$ on the entire interval $[0,T)$. We now seek to establish final control over the $H^3$-norms. Note that we already have estimate (6.13) which with the new information readily implies

$$\partial_t |e''|_2^2 \lesssim |e''|_2^2 + |u'''|_2^2.$$

Now we get to bounds on $|u'''|_2^2$. Not surprisingly all of the estimates mimic the already obtained sharper estimates for $H^2$ with the use of dissipation. In what follows we will indicate necessary changes and refer to appropriate places in Section 6.3 for details. Also, we will drop from the estimates all quantities that are already known to be bounded, such as $|u,\rho|_{H^2}$, etc. Thus, following (6.4) we can see that all the terms on the left hand side obey the bound by $|u'|_\infty |u'''|_2^2 \lesssim |u'''|_2^2$. We are left with the four terms on the right-hand side:

$$\mathcal{T}(\rho''', u)u''', \quad \mathcal{T}(\rho'', u')u''', \quad \mathcal{T}(\rho', u'')u''', \quad \mathcal{T}(\rho, u''')u'''.$$

First, the dissipation term obeys the same bound (6.8) where we now keep the dissipation :

$$\int \mathcal{T}(\rho, u''')u''' \,\mathrm{d}x \leqslant -\rho_-|u'''|^2_{H^{1/2}} + C|u'''|_2^2(|\rho|_\infty + |\rho'|_\infty) \lesssim -\rho_-|u'''|^2_{H^{1/2}} + C|u'''|_2^2. \tag{6.27}$$

Next, the term $\mathcal{T}(\rho''', u)u'''$ will be estimated in the same way as (6.26) with replacements $\rho'' \to \rho'''$, $u'' \to u'''$. We have the bound

$$|\mathcal{T}(\rho''', u)u'''| \leqslant |\rho'''|_2|u'''|_2|\Lambda u|_\infty + \frac{\rho_-}{10}|u'''|^2_{H^{1/2}} + \frac{10}{\rho_-}|\rho'''|_2^{3/2}|u'''|_2^{1/2}.$$

Since $|\Lambda u|_\infty \leqslant |u|_{H^2} < C$ and $|\rho'''|_2 \leqslant |e''|_2 + |u'''|_2$ we have

$$|\mathcal{T}(\rho''', u)u'''| \leqslant |e''|_2^2 + |u'''|_2^2 + \frac{\rho_-}{10}|u'''|^2_{H^{1/2}}.$$

Next, the term $\mathcal{T}(\rho'', u')u'''$ will also be estimated as in (6.26) with a simple replacement $u \to u'$, i.e., raising the derivative of $u$ by one on every step. We obtain directly,

$$|\mathcal{T}(\rho'', u')u'''| \leqslant |\rho''|_2|u'''|_2|\Lambda u'|_\infty + \varepsilon|u'''|^2_{H^{1/2}} + \frac{1}{\varepsilon}|\rho''|_2^{3/2}|u'''|_2^{1/2}|u''|_\infty.$$

Dropping $|\rho''|_2$ and using that $|\Lambda u'|_\infty, |u''|_\infty \leqslant |u|_{H^3}$, we obtain

$$|\mathcal{T}(\rho'', u')u'''| \leqslant \frac{\rho_-}{10}|u'''|^2_{H^{1/2}} + |u|^2_{H^3} \leqslant \frac{\rho_-}{10}|u'''|^2_{H^{1/2}} + C + |u'''|_2^2.$$

Finally, the term $\mathcal{T}(\rho', u'')u'''$ can be estimates as term (6.24) by raising the derivative of $u$ by one and with the use of boundedness of $|\rho''|_2$, $|\rho'|_\infty$. We obtain

$$\begin{aligned}|\mathcal{T}(\rho', u'')u'''| &\leqslant \varepsilon|u'''|^2_{H^{1/2}} + \frac{1}{\varepsilon}|\rho''|_2^{3/2}|u''|_4|u''|_\infty + |\rho''|_2^{1/2}|u'''|_2^{3/2} \\ &\quad + \left|\int u''(x)H\rho'(x)u'''(x)\,\mathrm{d}x\right| + |\rho'|_\infty|u'''|_2^2.\end{aligned}$$

We have trivially, $|u''|_4|u''|_\infty \leqslant |u|^2_{H^3}$, and

$$\left|\int u''(x)H\rho'(x)u'''(x)\,\mathrm{d}x\right| \leqslant |u''|_2|u'''|_2|H\rho'|_\infty \lesssim |u'''|_2|H\rho''|_2 \lesssim |u'''|_2.$$

This completes the estimate for the $H^3$-norm $Y$: $Y' \leqslant CY$ on the time interval of existence. This completes the proof.

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# ERRATUM: EULERIAN DYNAMICS WITH A COMMUTATOR FORCING

ROMAN SHVYDKOY AND EITAN TADMOR

The publication [1] has a minor gap in the argument presented in Section 6.2 where the authors establish control over the first derivatives of density and momentum. Specifically, the bound on $\Lambda\rho$ used in the momentum equation involves term $\sqrt{D\rho'(x)}$, which propagates into formula (6.21). At that point the authors combined (6.21) with (6.19) to get rid of the $D$-term. The mistakes presents in the fact that the point $x$ at which the $D$-term is evaluated in 6.19 is different from the point $x$ at which it is evaluated in 6.21. Hence the values may be different.

To avoid using combination of 6.19 and 6.21 we argue as follows. We produce a uniform bound on $|\rho''|_2$ on the time interval in question. This uniform bound, by Sobolev embedding, implies that $\rho' \in C^{\frac{1}{2}}$ uniformly. Then the trivial bound

$$|\Lambda\rho|_\infty \leqslant |\rho'|_{C^{1/2}},$$

implies uniform control over $\Lambda\rho$. Hence it is not necessary to resort to 6.19 to contain $\Lambda\rho$, and the rest of the estimates on $m'$ follow as documented in [1].

To achieve uniform bound on $|\rho''|_2$ we differentiate the density equation twice:

$$\partial_t\rho'' + u\rho''' + u'\rho'' + e''\rho + 3e'\rho' + 2e\rho'' = -2\rho''\Lambda\rho - 3\rho'\Lambda\rho' - \rho\Lambda\rho''.$$

Using that $u' = e + \Lambda\rho$, we obtain

$$\partial_t\rho'' + u\rho''' + e''\rho + 3e'\rho' + 3e\rho'' = -3\rho''\Lambda\rho - 3\rho'\Lambda\rho' - \rho\Lambda\rho''.$$

At this point we know that $|e^{(k)}| \lesssim \rho^{(k)}$, and we have uniform bounds on $\rho, \rho'$. So, testing with $\rho''$, integrating by parts in $u\rho'''\rho''$ term, and using the $e$ quantity again, we obtain

$$\partial_t|\rho''|_2^2 \lesssim |\rho''|_2 + |\rho''|_2^2 + |\Lambda\rho|_\infty|\rho''|_2^2 + |\rho''|_2|\Lambda\rho'|_2 - \int_{\mathbb{T}} \rho\rho''\Lambda\rho'' dx.$$

Using that $|\Lambda\rho'|_2 \lesssim |\rho''|_2$, and log-Sobolev inequality

$$|\Lambda\rho|_\infty \leqslant |\rho'|_\infty(1 + \log_+ |\rho''|_2) \lesssim 1 + \log_+ |\rho''|_2,$$

we further obtain

$$\partial_t|\rho''|_2^2 \lesssim C + |\rho''|_2^2(1 + \log_+ |\rho''|_2) - \int_{\mathbb{T}} \rho\rho''\Lambda\rho'' dx.$$

Using symmetrization in the remaining dissipation term we have

$$-\int_{\mathbb{T}} \rho\rho''\Lambda\rho'' dx = -\int_{\mathbb{T}} \rho D\rho'' dx + R, \tag{0.1}$$

where

$$R = \int_{\mathbb{T}} \rho''(x) \int_{\mathbb{T}} \frac{(\rho(x) - \rho(y))(\rho''(x) - \rho''(y))}{|x-y|^2} dy dx.$$

*Date*: October 21, 2018.

Using the bound $|\rho'| < C$ we further conclude

$$|R| \lesssim \int_{\mathbb{T}} |\rho''(x)| \int_{\mathbb{T}} \frac{|\rho''(x) - \rho''(y)|}{|x-y|} dy dx \leqslant \int_{\mathbb{T}} |\rho''(x)| \sqrt{D\rho''} dx \leqslant |\rho''|_2 \sqrt{\int_{\mathbb{T}} D\rho'' dx}.$$

By Young, the latter is bounded by

$$|R| \leqslant \varepsilon \int_{\mathbb{T}} D\rho'' dx + C_\varepsilon |\rho''|_2^2,$$

where $\varepsilon$ is smaller than the lower bound on the density on the given time interval. This gets the $D$-term absorbed into dissipation term in (0.1). We thus arrive at

$$\partial_t |\rho''|_2^2 \lesssim C + |\rho''|_2^2 (1 + \log_+ |\rho''|_2).$$

The result follows by integration.

Since in the estimates above we relied on second order a priori bound $|e''| \lesssim |\rho''|$ it is necessary to raise the regularity class from $H^3$ as in [1] to $H^4$ so that the local transport equation for $e''$ can be solved classically. The idea to avoid using higher order a priori bounds $|e^{(k)}| \lesssim |\rho^{(k)}|$ is to abandon the use of momentum equation for $m$, where $e$ quantity is explicitly present, and instead come back to the $u$-equation. This was performed in [1] up to the order 3 space $H^4$, and the argument is entirely similar going one more derivative up to $H^4$. We therefore state our final result as follows.

**Theorem 0.1.** *Consider the system of equations (1.1),* [1], *with* $1 \leqslant \alpha < 2$ *subject to initial data* $(u_0, \rho_0) \in H^4(\mathbb{T}^1) \times H^{3+\alpha}(\mathbb{T}^1)$. *Then the system admits a global solution in the same class.*

## References

[1] R. Shvydkoy and E. Tadmor, *Eulerian dynamics with a commutator forcing*, Trans. Math. and Appl. 1(1) (2017) 1-26.

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